MA 441, Fall 2013

Preparation for the Final.

Below you can find a list of definitions, axioms (well, an axiom), theorems and (counter-)examples that you need to know for the Final. More precisely, for in-class part of the Final, you will be given several (tentatively, ten to twenty) items from this list to formulate. Note that it won't have to be word-by-word citation, but whatever you write will need to be (a) correct (as in not a false statement), (b) easily equivalent to the textbook/lectures version.

On in-class part of the Final, you will *not* be asked to provide any proofs.

Knowledge of proofs of statements on this list is required for the take-home part of the Final.

In the list below,
⊙ marks definitions and an axiom;
□ marks theorems and statements;
▷ marks (counter-)examples that you need to know off-hand.

Note that there may be corrections to this list until Tuesday, Dec 10.

Preliminaries.

- ⊙ Sets, set-theoretic operations, functions, inverse function, composition.
- ⊙ Injections, surjections, bijections.
- ⊙ Finite, infinite sets. Denumerable (countably infinite), countable, uncountable sets.
- □ Countability of $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z}^n$, $\mathbb{Q}$.
- □ Cantor's Theorem. Uncountability of $\mathbb{R}$.

Properties of $\mathbb{R}$.

- ⊙ Arithmetic properties of $\mathbb{R}$ (A1-A4, M1-M4, D). (You will be provided a reminder sheet with for this item, if it will be relevant on the Final.)
- □ Basic consequences of arithmetic properties of $\mathbb{R}$.
- ⊙ Order properties of $\mathbb{R}$: set of positive elements, defining order in terms of the set of positive elements. (You will be provided a reminder sheet with for this item, if it will be relevant on the Final.)
- □ Basic properties of order on $\mathbb{R}$. (You will be provided a reminder sheet with for this item, if it will be relevant on the Final.)
- □ Positivity of squares and other basic consequences of properties of order on $\mathbb{R}$.
- □ Triangle inequality.
- ⊙ Bounded, bounded above, bounded below subsets of $\mathbb{R}$.
- ⊙ Upper bound, lower bound of a subset of $\mathbb{R}$.
- ⊙ Least upper bound (= exact upper bound = supremum) of a subset of $\mathbb{R}$.
- ⊙ Greatest lower bound (= exact lower bound = infimum) of a subset of $\mathbb{R}$.
- ⊙ Completeness property of $\mathbb{R}$ (= supremum property of $\mathbb{R}$).

- ☐ Archimedean property of $\mathbb{R}$.
- ☐ Nested intervals property.
- ☐ The density theorem.

Limits of Sequences.

- ⊙ Sequence of real numbers (= sequence in $\mathbb{R}$).
- ⊙ Limit of a sequence in $\mathbb{R}$, convergent/divergent sequence.
- ☐ Uniqueness of limit of a sequence.
- ⊙ Tail of a sequence.
- ⊙ Bounded sequence.
- ☐ Boundedness of a convergent sequence.
- ▷ Bounded but divergent sequence.
- ☐ Arithmetic properties of limits of sequences (Theorem 3.2.3).
- ▷ Divergent sequences A, B such that $A + B$ converges.
- ☐ Order properties of limits of sequences (Theorems 3.2.4, 3.2.5).
- ▷ Sequence (a_n) with $a_n > 0$ for all $n \in \mathbb{N}$, but $\lim(a_n) = 0$.
- ☐ Squeeze theorem for sequences.
- ☐ Increasing, strictly increasing, decreasing, strictly decreasing, monotone sequence.
- ☐ Monotone convergence theorem.
- ⊙ Euler's number e.
- ☐ Square root as a limit of a sequence.
- ⊙ Subsequence of a sequence.
- ☐ Bolzano–Weierstrass theorem.
- ⊙ Cauchy sequence.
- ☐ Cauchy criterion.
- ⊙ Sequence that tends to $+\infty$, sequence that tends to $-\infty$, properly divergent sequence.
- ⊙ Sum of a series. Correspondence between infinite series and sequences.
- ☐ n-th term test.

Limits of Functions.

- ⊙ Cluster point of a subset of $\mathbb{R}$.
- ⊙ Limit of a function.
- ☐ Uniqueness of limit of a function.
- ☐ Sequential criterion for limit of a function.
- ⊙ Function bounded a neighborhood.
- ☐ Local boundedness of a function that has a limit.
- ▷ Bounded function that does not have a limit at 0.

- ☐ Arithmetic properties of limits of functions (Theorem 4.2.4).
- ▷ Functions f, g that don't have a limit at some point $c \in \mathbb{R}$, but $f + g$ does.
- ☐ Order properties of limits of functions (Theorem 4.2.6).
- ▷ Functions f, g such that for all x in their domain, $f > g$, but at some point c, $\lim_{x \to c} f = \lim_{x \to c} g$.
- ☐ Squeeze theorem for limits of functions.
- ☐ Local separation from zero (Theorem 4.2.9).
- ⊙ Infinite limit of a function, limit of a function at infinity, infinite limit of a function at infinity.
- ⊙ One-sided limits.

Continuous Functions.

- ⊙ Function, continuous at a point. Function, discontinuous at a point.
- ☐ Criterion for continuity in terms of neighborhoods (Theorem 5.1.2).
- ☐ Sequential criterion for continuity.
- ☐ Sequential criterion for discontinuity.
- ⊙ Function, continuous on a subset of $\mathbb{R}$.
- ▷ Function, discontinuous everywhere (for example, Dirichlet's function).
- ▷ Function, continuous at irrational numbers and discontinuous at rational numbers (for example, Thomae's function).
- ☐ Arithmetic properties of continuous functions (Theorem 5.2.1).
- ▷ Functions f, g discontinuous at 0 such that $f + g$ is continuous at 0.
- ☐ Composition of continuous functions (at a point and on a set).
- ☐ Boundedness Theorem.
- ▷ Bounded but discontinuous (at least at one point) function.
- ▷ Function continuous but unbounded on an open interval.
- ⊙ Absolute (= global) maximum of a function on a set, point of absolute maximum. Absolute minimum of a function on a set, point of absolute minimum.
- ☐ Maximum–Minimum Theorem.
- ▷ Function f continuous on an open interval such that that f does not have maximum or minimum value.
- ☐ Location of roots theorem, Bolzano's intermediate value theorem.
- ☐ Preservation of closed intervals. Preservation of intervals.
- ⊙ Function, uniformly continuous on a subset of $\mathbb{R}$.
- ☐ Nonuniform continuity criterion.
- ☐ Uniform continuity theorem.
- ▷ Function, continuous but not uniformly continuous on an open interval.

- ⊙ Increasing, strictly increasing, decreasing, strictly decreasing, monotone functions.
- ⊙ Jump of a monotone function.
- ☐ Continuity criterion of monotone functions (Theorem 5.6.3).
- ☐ Continuous inverse theorem.

Differentiation.

- ⊙ Derivative of a function at a point. Function, differentiable at a point.
- ☐ Continuity of a differentiable function.
- ▷ Function, continuous but not differentiable at $x = 0$.
- ▷ Function, differentiable on $\mathbb{R}$, whose derivative is not continuous at 0.
- ☐ Arithmetic properties of derivative.
- ☐ Chain rule.
- ☐ Derivative of inverse function (inverse function theorem).
- ☐ Interior extremum theorem.
- ☐ Rolle's theorem.
- ☐ Mean value theorem.
- ☐ First derivative test for extrema (Theorem 6.2.8).
- ☐ Criterion for a differentiable function to be increasing/decreasing/constant on an interval (Theorems 6.2.5, 6.2.7).
- ⊙ nth Taylor polynomial of a function.
- ☐ Taylor's theorem.
- ☐ nth derivative test for extrema (Theorem 6.4.4).
- ☐ nth Taylor's polynomial at zero for $(1 + x)^\alpha$, e^x, $\ln x$, $\sin x$, $\cos x$.

The Riemann Integral.

- ⊙ Partition, tagged partition, Riemann sum.
- ⊙ Riemann integrable function, Riemann integral.
- ▷ Bounded but not a Riemann integrable function.
- ☐ Arithmetic and order properties of Riemann integral (linearity and monotonicity).
- ☐ Boundedness theorem for Riemann integrable function.
- ☐ Riemann integrability of a continuous function, monotone function (proof not required).
- ☐ Interval additivity theorem (proof not required).
- ☐ The fundamental theorem of calculus (first form).
- ⊙ Indefinite integral, antiderivative.
- ☐ The fundamental theorem of calculus (second form).
- ☐ Derivative of an indefinite integral of a continuous function.